Scientific report

Regarding the implementation of the project for the period October – December 2011

The team of the *'Hopf algebras and related topics'* project:

Prof. dr. Gigel Militaru -- CS III dr. Sebastian Burciu: hired on 8.12.2011 -- AC, PhD student Costel Bontea: hired on 8.12.2011 -- AC, PhD student Ana Agore (on leave at Free Univ. Brussel until June 30th 2012).

Papers accepted for publication:

[AM1] A. L. Agore and G. Militaru, *Schreier type theorems for bicrossed products,* accepted for publication at 17. 10. 2011 in Central European J. Math. (ISI – IF 2010: 0.581)

Papers sent for publication:

[AM2] A. L. Agore and G. Militaru, *Extending Structures I: the level of groups.*

[AM3] A. L. Agore and G. Militaru, *Unified products and split extensions of Hopf algebras*.

[ACM4] A.L. Agore, S. Caenepeel, G. Militaru, *Braidings on the category of bimodules, Azumaya algebras and epimorphisms of rings.*

[ACM5] A.L. Agore, S. Caenepeel, G. Militaru, *The center of the category of bimodules and descent data for non-commutative rings*.

Papers in final stage:

[SB1] S. Burciu, "*Mueger centralizers for the category of representations of a semisimple Drinfeld double*".

The summary of the results obtained so far:

The paper [AM1] represents the first step in the approach for solving **problem (1b)** of the projects **first objective**. At the group level, we have completely classified the bicrossed products of two groups by describing, up to an isomorphism, all the groups E that *factorize* through two fixed groups H and G. We have proved that this classification is parameterized by a combinatorial data set consisting of an automorphism of H, a permutation of G and a transition map v between G and H. As an application of our results, we prove two Schreier type theorems regarding the classification and we present in detail some examples for cyclic groups. The results of [AM1] can serve as a source of inspiration for attacking the factorization problem in other categories of mathematical objects: Lie algebras, Lie groups, quantum groups, etc. At the Hopf algebra level we have already started this study together with A. Agore and C. Bontea (PhD students in the project team) and the paper is being edited so it can be sent for publication.

The paper [AM2] is the first paper in a long series of other papers in which we have initiated *the extending structures problem* (TESP) – this is **problem (1a) of the first objective**. TESP unifies two well known problems from group theory (*the extension problem* stated by Holder and *the factorization problem* stated by Ore) and can be formulated in many areas of mathematics: groups, as in this paper, Lie algebras, Lie groups, algebras, coalgebras, Hopf algebras, locally compact groups or locally compact quantum groups, etc. At the group level, TESP has a very simple and attractive statement: *let H be a group and let E be a set which contains H. Describe and classify up to an isomorphism that stabilizes H all the group structures * that can be defined on E such that H is a subgroup of (E, *)*. We have given a complete answer to this question in [AM2]: for this we have introduced a new product for groups, which we named the *unified product*, and which contains as special cases Schreier's crossed product and Takeuchi's bicrossed product. The unified product is associated to a group and to a new algebraic structure (S, *), where * is a multiplication on S (S is a fiber in 1 of a surjection) which admits a unit, every element of S is left invertible and the associativity condition on S is deformed by a cocycle f and a right action of H on S. We have proved that any group structure on the set E such that H is a subgroup of E is isomorphic to a unified product and we have classified up to an isomorphism which stabilizes H all the unified products. These are classified by a cohomological set H^2 (H, S) which plays the rôle of the second cohomology group for Holder's extension problem. As corollaries, we obtain a Schreier type theorem and we give an answer to a recent problem posed by Kuperberg. We mention that this construction is also intimately related to a well known and of interest problem in differential geometry: the existence of *hidden symmetries* for a principal H-fibre (H is a Lie group), to which we give an answer in the case of 0-dimensional varieties, i.e. discrete sets. The ideas contained in the paper [AM2] have great potential of being developed in other areas of mathematics and these have been explained in detailed in the last section of the paper.

[AM3] is a paper dealing also with **problem (1a) of the first objective**.Having [AM1] as a basis, we have constructed the unified product at the level of quantum groups in the paper [A.L. Agore, G. Militaru]: *Extending Structures II: The Quantum Version,* Journal of Algebra, Vol. 336 (2011), 321 – 341. Here the unified product for Hopf algebras is constructed from the factorization viewpoint: a Hopf algebra E is isomorphic to a unified product between A and H if and only if E factorizes through the Hopf subalgebra A and the subcoalgebra H. [AM3] approaches the unified product from the dual viewpoint, the one of split extensions of Hopf algebras. More precisely, we have proven a new structural theorem: a Hopf algebra E is isomorphic to a unified product between A and H if and only if there exists a Hopf algebra morphism i: A → E, that admits a retract which is a left A-linear coalgebra map and is normal in the sense of Andruskiewitsch and Devoto. In fact, if we view it at a categorical level, the unified product gives an answer to a natural problem: that of completely describing the split monomorphisms, in a relative sense, in a given category C. The link between the unified product for Hopf algebras and the well-known Radford biproduct is demonstrated by giving a necessary and sufficient criterion for the natural extension of the unified product to split as a Hopf algebra morphism. As a consequence, a new method of constructing unified products is proposed and a concrete example is constructed using a minimal set of data: a group, a pointed set on which the group acts and a *transition* map between the group and the pointed set.

The papers [ACM4] and [ACM5] give the first answers to **problem (2b)** of the project's **second objective**. If I was to make a hierarchy of the most beautiful results obtained in over 20 years of research then the main theorem of [ACM4] would surely be among the first three. An explanation: noncommutative algebra was born from the study of finite dimensional central simple algebras having Frobenius, Wedderburn, Artin, Albert, Brauer and others as parents. With the introduction by Joyal and Street of braided categories, the theory of quantum groups experienced a kind of revolution, owing to the unifying character of braided categories and their deep implications in physics, knot theory, Kac-Moody algebras or quantum field theories, etc. The main result of [ACM4] establishes a surprising link between the classical concept of central simple algebra from the beginnings of the subject of algebra and the modern concept of braided category. More precisely, we have proven the following theorem: a finite dimensional algebra A is central simple if and only if the monoidal category of A-bimodules is braided. Furthermore, the result is equivalent to the existence of a (unique) element R of A \otimes A \otimes A satisfying certain natural and fairly elementary axioms. We have named such a pair (A, R) a quasitriangular algebra. The result was stated at a more general level by working over commutative rings – in this case, the concept of a central simple algebra is replaced by the concept of an Azumaya algebra. The existence of a braiding (if it exists then it is unique and is a symmetry) on the category of A-bimodules is in itself a surprising fact because even the trivial flip map (m, n) → (n, m), which is the canonical braiding for k-modules for a commutative ring k, not only is not a braiding but it is not even well defined in the noncommutative case. The quasitriangular algebras (A, R) which we have introduced generalize the concepts of central simple algebras (in the finite dimensional case the two concepts are the same) and Azumaya algebras. The commutative case is also surprising: a commutative algebra is quasitriangular if and only if the algebra structural map is an epimorphism (in a categorical sense) and in this case R is trivial. For any quasitriangular algebra (A, R) we have constructed a new family of solutions to the well-known quantum Yang-Baxter equation. [ACM5] continues the previous work: the main theorem gives equivalent descriptions for the Drinfel'd center of the monoidal category of A-bimodules. From among these, we mention three which are special: the category of Grothendieck noncommutative descent, the category of corepresentations over Sweedler's canonical coring and the category of modules with flat connections from noncommutative differential geometry. Being the centers of a monoidal category all these categories are braided categories: therefore one can mimic in these categories most of the constructions which exist for differentiable manifolds.

During this stage of the project Sebastain Burciu worked on **problem (2e)** of the projects **second objective** concerning Mueger's centralizer for braided fusion categories. Specifically, he studied Mueger's centralizer for the category of representations of semisimple quantum doubles. The results obtained so far are part of a paper [SB1] "Mueger centralizers for the category of representations of a semisimple Drinfeld double" which will soon be sent for publication. Mueger's centralizer of a fusion subcategory D is defined as the fusion subcategory generated by the simple objects X of C that commute with every simple object Y of the subcategory D, i.e. the composition of the braidings on XY and YX is the identity. Mueger's centralizer is an essential tool in the proof of the fact that two fusion categories are Morita equivalent if and only if their Drinfeld centers are equivalent as braided categories. It is analogous to the notion of orthogonal space in the theory of metric groups (which admit a quadratic form q). For example, for a quadratic form q, Mueger's centralizer of a subgroup H coincides with the orthogonal subgroup of H relative to q. Using the notion of the centralizer introduced by Mueger, Bruguieres succeeded in giving necessary and sufficient conditions for a premodular braided category to admit a modularization. We recall that the notion of a modular category is an essential tool in the definition of conformal quantum field theory. Another important feature of Mueger's centralizer is that, similar to the case for quadratic groups, it is used in the decomposition of fusion categories as direct products of categories. At present, there is no general formula for the centralizer of a fusion category in the scholarly literature.

Let A be a factorizable semisimple Hopf algebra. Then the category of representations Rep(A) is a braided monoidal category. Moreover, for every fusion subcategory D of Rep(A), its Mueger centralizer satisfies D'' = D. A fusion subcategory of Rep(A) is, in general, of the form Rep(A//L) for a certain normal coideal subalgebra L of A. Using Mueger's centralizer, we obtain that Rep(A//L)' =Rep(A//L') for another normal coideal subalgebra L' of A. The main goal of this problem is to identify the link which exists between the coideal subalgebras L and L'.

The results obtained so far in [SB1] answer this question when A=D(H). It is known in this case that D(H) is factorizable and that the category Rep(D(H)) is a modular tensor category. Working in this setting we succeeded in identifying the coideal subalgebra L' in the case when L is a normal Hopf subalgebra of A. We show that L' must also be a normal Hopf subalgebra of D(H). On the other hand, the normal Hopf subalgebras of D(H) were recently classified by S. Burciu and they are of the form D(K,L,X, ψ) where K and L are normal Hopf subalgebras of H commuting with each other. If X and ψ are trivial then we use the short notation D(K,L) for D(K,L,1,1). In the preprint [SB1] we prove the following:

Theorem 1: With the previous notations we have Rep(A//D(K,H))'=Rep(A//D(H,K))

In other words, the above theorem implies that if L=D(K,H) then L'=D(H,K). We expect to have applications in the classification of modular fusion categories, as a consequence of Theorem 1. We mention that the result of Theorem 1 generalizes a recent result obtained by Naidu, Nikshych si Witherspon in IMRN 22/2009. This paper considers the case of the Drinfeld double D(G) of a finite group G. In this case the fusion subcategories of Rep(D(G)) are of the form D(K,H, λ), where K and H are normal subgroups of G commuting with each other, i.e. [K,H]=1, and λ is a bicharacter of the direct product K x H. The case of the other normal Hopf subalgebras D(K,L, X, ψ) of D(H), when X and ψ are not trivial, remains to be studied.

There has also been important progress made in the study of **problem (2d)** of the project's **second objective**. Together with A. Bruguieres we proved a theorem which characterizes the normal tensor functors in terms of the equivalence relations of the constituents. This equivalence relation is similar to the one given by Rieffel in the case of extensions of finite dimensional algebras. Together with S.Natale we started a work project in which, among other things, we study the forgetful functor C^G→ C. A formula of its adjoint functor Ind is given which allows for the description of all simple objects of C^G as well as for its fusion subcategories.

(5 pages max)

Project Director,

Prof. dr. Gigel Militaru