A new characterization of injective and surjective functions and group homomorphisms

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Abstract. A model of a function f between two non-empty sets is defined to be a factorization $f = \pi \circ i$, where π is a surjective function and i is an injective function. In this note we shall prove that a function f is injective (respectively surjective) if and only if it has a final (respectively initial) model. A similar result, for groups, is also proven.

Keywords: factorization of morphisms, injective/surjective maps, initial/final objects

MSC: Primary 03E20; Secondary 20A99

1 Introduction and preliminary remarks

In this paper we consider the sets taken into account to be non-empty and groups denoted differently to be disjoint. For basic concepts on sets and functions see [1]. It is a well known fact that any function $f\colon A \to B$ can be written as the composition of a surjective function followed by an injective function. Indeed, $f = f' \circ id_B$, where $f'\colon A \to Im(f)$ is the restriction of f to $Im(f)$ is one such factorization. Moreover, it is an elementary exercise to prove that this factorization is unique up to an isomorphism: i.e. if $f = i \circ \pi$, $i\colon A \to X$, $\pi\colon X \to B$ is another factorization for f, then we easily see that $X = i^{-1}\left(Im(f)\right)$ and that $\pi = i^{-1} \circ f'$. The purpose of this note is to study the dual problem. Let $f\colon A \to B$ be a function. We shall define a *model* of the function f to be a triple (X, i, π) made up of a set X, an injective function $i\colon A \to X$ and a surjective function $\pi\colon X \to B$ such that $f = \pi \circ i$, that is, the following diagram is commutative.

A model (X, i, π) for the function f is called an *initial model* if for any other model (Y, j, p) of the function f there is a unique function $g\colon Y \to X$ such that $g \circ j = i$ and $p \circ g = \pi$. A model (X, i, π) for the function f is called a *final model* if for any other model (Y, j, p) of the function f there is a unique function $g\colon X \to Y$ such that $g \circ i = j$ and $\pi \circ g = p$.

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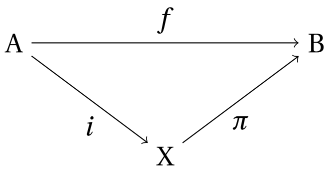


Remark 1.1. *For any function $f: A \to B$, there is at least one model.*

We may consider that A and B do not have any elements in common, otherwise, just consider a set $B' \cong B$. Indeed, it is easy to see the remark is true if we consider $X = A \cup B$, consider i to be the canonical injection and define

$$\pi(x) = \begin{cases} f(a), & \text{if } a \in A \\ b, & \text{if } b \in B \end{cases}.$$

2 Initial and final models of a function

The next theorem gives a new characterization of injective and surjective maps.

Theorem 2.1. *Let $f: A \to B$ be a function. Then:*

a) f has a final model if and only if f is injective.

b) f has an initial model if and only if f is surjective.

Proof. a) " $\Longleftarrow$ " Let f be an injective function. In this case, (B, f, id_B) is obviously a model for f and we will now show that it is a final model. Let (Y, j, p) be another model. We want there to be a unique function $g: Y \to B$ such that $g \circ j = f$ and $id_B \circ g = p$. The existence is obvious since $g = p$ satisfies the relations. The uniqueness is given by the fact that $id_B \circ g = p$, and so the unique function that satisfies this is $g = p$.

" $\Longrightarrow$ " Let $f: A \to B$, assume (X, i, π) is a final model of f and $\alpha \notin X$. Suppose f is not injective. Therefore, there are two elements of A, $a_1 \neq a_2$ such that $f(a_1) = f(a_2)$. Let $Y = X \cup \{\alpha\}$. We define $j: A \to Y$ and $p: Y \to B$ such that

$$j(a) = i(a) \quad \text{and} \quad p(x) = \begin{cases} \pi(x), & \text{if } y \in X \\ f(a_1), & \text{if } y = \alpha \end{cases},$$

$\forall a \in A$ and $\forall y \in Y$.

With this construction, (Y, j, p) is a model for f as $\forall a \in A$,

$$(p \circ j)(a) = p\big(j(a)\big) = p(i(a)) = \pi(i(a)) = f(a),$$

since $i(a) \in X$ and (X, i, π) is a model for f. We now define $g_1, g_2: Y \to X$ as

$$g_1(y) = \begin{cases} y, & \text{if } y \in X \\ i(a_1), & \text{if } y = \alpha \end{cases} \quad \text{and} \quad g_2(y) = \begin{cases} y, & \text{if } y \in X \\ i(a_2), & \text{if } y = \alpha \end{cases}.$$

Then $g_1 \neq g_2$, since $i(a_1) \neq i(a_2)$ because i is injective. We are now going to prove that g_1 and g_2 are morphisms of models (i. e. the triangles are commutative) which contradicts the hypothesis that X is a final model (X is final $\implies g_1 = g_2$). Firstly,

$$(g_1 \circ j)(a) = g_1\big(j(a)\big) = g_1\big(\underbrace{i(a)}_{\in X}\big) = i(a),$$

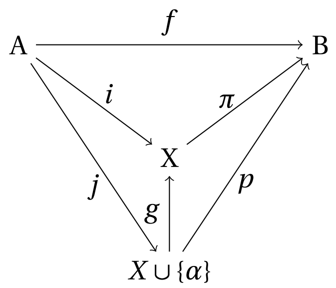
$\forall a \in A$, which means that $g_1 \circ j = i$ and, analogously, $g_2 \circ j = i$. Now, for $y \in Y$ we have

$$(\pi \circ g_1)(y) = \pi\big(g_1(y)\big) = \begin{cases} \pi(y), & \text{if } y \in X \\ \pi\left(i(a_1)\right), & \text{if } y = \alpha \end{cases} = \begin{cases} \pi(y), & \text{if } y \in X \\ f(a_1), & \text{if } y = \alpha \end{cases} = p(y)$$

and, analogously,

$$(\pi \circ g_2)(y) = \pi\big(g_2(y)\big) = \begin{cases} \pi(y), & \text{if } y \in X \\ \pi\left(i(a_2)\right), & \text{if } y = \alpha \end{cases} = \begin{cases} \pi(y), & \text{if } y \in X \\ f(a_2), & \text{if } y = \alpha \end{cases} = p(y).$$

In conclusion, $g_1 \neq g_2$ and both $g = g_1$ and $g = g_2$ commutatively close the diagram



which contradicts the fact that (X, i, π) is a final model. The proof is now complete.

b) " $\Longleftarrow$ " Let f be an surjective function. In this case, (A, id_A, f) is obviously a model for f and we will now show that it is an initial model. Let (Y, j, p) be another model. We want there to be a unique function $g\colon A \to Y$ such that $id_A \circ g = j$ and $p \circ g = f$. The existence is obvious since $g = j$ satisfies the relations. The uniqueness is given by the fact that $id_A \circ g = j$, and so the unique function that satisfies this is $g = j$.

" $\implies$ " Let $f\colon A \to B$, assume (X, i, π) is an initial model of f and $\{-\infty, \infty\}$ two different elements which are not part of A, B or X. Suppose f is not surjective. Therefore, there exists $\alpha \in X \setminus i(A)$ (otherwise, i is surjective, and so bijective and $f = \pi \circ i$ which implies that f is a composition of surjective functions so it is surjective, contradiction). Let $Y = X \setminus \{\alpha\} \cup \{-\infty, \infty\}$. We define $j\colon A \to Y$ and $p\colon Y \to B$ such that

$$j(a) = i(a) \in X \quad \text{and} \quad p(y) = \begin{cases} \pi(y), & \text{if } y \in X \setminus \{\alpha\} \\ \pi(\alpha), & \text{if } y \in \{-\infty, \infty\} \end{cases},$$

$\forall a \in A$ and $\forall y \in Y$.

Obviously, j is an injective function and p is surjective because

$$p(Y) = \pi(X \smallsetminus \{\alpha\}) \cup \{\pi(\alpha)\} = \pi(X) = B.$$

Moreover,

$$(p \circ g)(a) = p\big(j(a)\big) = p(\underbrace{i(a)}_{\in X \smallsetminus \{\alpha\}}) = \pi(i(a)) = f(a),$$

so (Y, j, p) is a model for f. We now define $g_1, g_2 \colon X \to Y$ to be the functions

$$g_1(x) = \begin{cases} x, & \text{if } x \in X \smallsetminus \{\alpha\} \\ -\infty, & \text{if } x = \alpha \end{cases} \quad \text{and} \quad g_2(x) = \begin{cases} x, & \text{if } x \in X \smallsetminus \{\alpha\} \\ \infty, & \text{if } x = \alpha \end{cases}.$$

It is straightforward that $g_1 \neq g_2$, since $\infty \neq -\infty$. We shall now prove that g_1 and g_2 are morphisms of models which contradicts the fact that (X, i, π) is an initial model (X is initial $\implies g_1 = g_2$). First of all,

$$(g_1 \circ i)(a) = g_1(\underbrace{i(a)}_{\in X \smallsetminus \{\alpha\}}) = i(a) = j(a), \forall a \in A,$$

so $g_1 \circ i = j$ and, in an analogous way, $g_2 \circ i = j$. Now, for $p \circ g_1 = p \circ g_2 = \pi$ we have to split our analysis into two parts:

- If $x \in X \smallsetminus \{\alpha\}$, we have

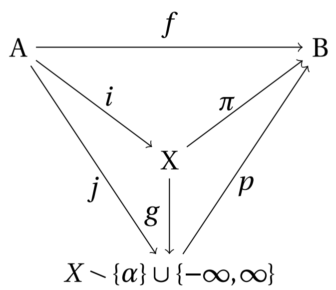
$$g_1(x) = g_2(x) = x \implies p\big(g_1(x)\big) = p\big(g_2(x)\big) = p(x) = \pi(x), \forall x \in X \smallsetminus \{\alpha\}.$$

- If $x = \alpha$, we have $g_1(\alpha) = -\infty$ and $g_2(\alpha) = \infty$, so

$$p\big(g_1(\alpha)\big) = p(-\infty) = \pi(\alpha) = p(\infty) = p\big(g_2(\alpha)\big)$$

  which shows that $p \circ g_1 = p \circ g_2 = \pi$.

In conclusion, $g_1 \neq g_2$ and both $g = g_1$ and $g = g_2$ commutatively close the diagram



which contradicts the fact that (X, i, π) is an initial model. The proof is now complete.

□

3 Generalization to groups

The term model used for sets can be generalized to groups in a natural way, by considering the functions to be group homomorphisms and the sets to be groups. The same problem is addressed in this case. We shall use the following definitions:

Definition 3.1. *Let G and H be two groups. We define the free product of G and H, denoted by $G \star H$, to be the group made up of all the finite words formed with elements of G and H in the reduced form.*

$$G \star H = \{g_1 h_1 g_2 h_2 \ldots | g_i \in G, h_i \in H\}$$

In terms of group presentations, if $G = (S_G | R_G)$ and $H = (S_H | R_H)$, this definition is equivalent to

$$G \star H = (S_G \cup S_H | R_G \cup R_H).$$

The notations used are standard, see [2, Chapter 11].

Definition 3.2. *Let G and H be two groups with $G \cap H = K$. We define the amalgamated free product of G and H with subgroup K, denoted by $G \star_K H$ to be the group $G \star H / N$, where N is the normal closure of K in $G \star H$.*

For basic concepts on amalgamated free products see [3, Chapter I.11.].

Remark 3.1. *There is at least one model for any homomorphism f.*

Indeed, let $f\colon A \to B$ be a group homomorphism. Consider $X = A \star B$ (see Definition 3.1.) and define $i\colon A \to X$ and $\pi\colon X \to B$ such that

$$i(a) = a \quad \text{and} \quad \pi(a_1 b_1 a_2 b_2 \ldots) = f(a_1) b_1 f(a_2) b_2 \ldots,$$

$\forall a \in A$ and $\forall a_i \in A$, $b_i \in B$.

It is obvious that both i and π are group homomorphisms. Since $\pi(b) = b, \forall b \in B$, π is surjective, i is injective by definition and $\pi(i(a)) = \pi(a) = f(a)$. Therefore, $\pi \circ i = f$. We have thus proved that any group homomorphism f has at least one model.

Theorem 3.1. *Let $f\colon A \to B$ be a group homomorphism. Then:*

a) f has a final model if and only if f is injective.

b) f has an initial model if and only if f is surjective.

Proof. a) " $\Longleftarrow$ " Let f be an injective group homomorphism. In this case, (B, f, id_B) is obviously a model for f and the fact that it is final is trivial and analogous to the proof for sets in Theorem 2.1. a).

" $\Longrightarrow$ " Let $f\colon A \to B$ be a homomorphism and assume (X, i, π) is a final model of f. Suppose f is not injective. Therefore, $Ker(f) \neq \{e_A\}$, so there is an element $a \in A$, $a \neq e_A$ for which $f(a) = e_B$. Let $Y = X \star < a >$ (see Definition 3.1.). We define $j\colon A \to Y$ and $p\colon Y \to B$ such that

$$j(a) = i(a) \quad \text{and} \quad p(x_1 a_{k_1} x_2 a_{k_2} \ldots) = \pi(x_1 x_2 \ldots),$$

$\forall a \in A$ and $\forall x_i \in X$.

With this construction, (Y, j, p) is a model for f as $\forall a \in A$,

$$(p \circ j)(a) = p\big(j(a)\big) = p\big(\underbrace{i(a)}_{\in X}\big) = \pi(i(a)) = f(a),$$

since (X, i, π) is a model for f.

We now define $g_1, g_2 \colon Y \to X$ as

$$g_1(x_1 a^{k_1} x_2 a^{k_2} \dots) = x_1 x_2 \dots \quad \text{and} \quad g_2(x_1 a^{k_1} x_2 a^{k_2} \dots) = x_1\,(i(a))^{k_1}\, x_2\,(i(a))^{k_2} \dots .$$

First of all, from the construction it is obvious that both g_1 and g_2 are homomorphisms. Indeed, the only non-trivial fact is:

$$g_1\left(\left(x_1 a^{k_1} x_2 a^{k_2} \dots\right)^{-1}\right) = g_1\left(\dots a^{-k_2} x_2^{-1} a_{-k_1} x_1^{-1}\right) = \dots x_2^{-1} x_1^{-1} = \left(g_1\left(x_1 a^{k_1} x_2 a^{k_2} \dots\right)\right)^{-1}.$$

The proof for g_2 follows the same steps. Then $g_1 \neq g_2$, since $i(a) \neq e_X$ because i is injective (which means that $Ker(i) = e_A$). We are now going to prove that g_1 and g_2 are homomorphisms of models (i. e. the triangles are commutative) which contradicts the hypothesis that X is a final model (X is final $\implies g_1 = g_2$). Obviously, $g_1 \circ j = g_2 \circ j = i$. Now, since

$$\pi\left(g_1(x_1 a^{k_1} x_2 a^{k_2} \dots)\right) = \pi(x_1 x_2 \dots) = p(x_1 a^{k_1} x_2 a^{k_2} \dots)$$

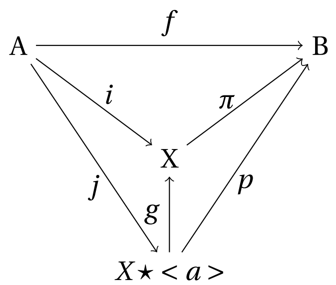
and

$$\pi\left(g_2(x_1 a^{k_1} x_2 a^{k_2} \dots)\right) = \pi(x_1\,(i(a))^{k_1}\, x_2\,(i(a))^{k_2} \dots) = \pi(x_1)\pi\left((i(a))^{k_1}\right)\pi(x_2)\pi\left((i(a))^{k_2}\right)\dots =$$

$$= \pi(x_1)\pi(x_2)\dots = \pi(x_1 x_2 \dots) = p(x_1 a^{k_1} x_2 a^{k_2} \dots),$$

we have proved that $g_1 \circ j = g_2 \circ j = i$ and $\pi \circ g_1 = \pi \circ g_2 = p$.

In conclusion, $g_1 \neq g_2$ and both $g = g_1$ and $g = g_2$ commutatively close the diagram



which contradicts the fact that (X, i, π) is a final model. The proof is now complete.

b) " $\Longleftarrow$ " Let f be a surjective group homomorphism. In this case, (A, id_A, f) is obviously a model for f and and the fact that it is initial is trivial and analogous to the proof for sets in Theorem 2.1 b).

" $\Longrightarrow$ " Let $f\colon A \to B$ be a homomorphism and assume (X, i, π) is an initial model of f. Suppose f is not surjective. Therefore, $X \smallsetminus i(A) \neq \emptyset$. Consider X' a group such that $X' \cap X = i(A)$ and $X' \cong X$ through the isomorphism ϕ which keeps $i(A)$ fixed. Let Y be the amalgamated free product of the groups X and X' with subgroup $i(A)$, i.e. $Y = X \star_{i(A)} X'$ (see Definition 3.2). We define $j\colon A \to Y$ and $p\colon Y \to B$ such that

$$j(a) = i(a) \quad \text{and} \quad p(x_1 x_1' x_2 x_2' \ldots) = \pi\left(x_1 \phi^{-1}(x_1') x_2 \phi^{-1}(x_2') \ldots\right),$$

$\forall a \in A$ and $\forall x_i \in X$, $\forall x_i' \in X'$.

With this construction, (Y, j, p) is a model for f as $\forall a \in A$,

$$(p \circ j)(a) = p\big(j(a)\big) = p\big(\underbrace{i(a)}_{\in X}\big) = \pi(i(a)) = f(a),$$

since (X, i, π) is a model for f and $p(X) = \pi(X) = B$.

We now define $g_1, g_2\colon Y \to X$ as

$$g_1(x) = x \quad \text{and} \quad g_2(x) = \phi(x),$$

$\forall x \in X$.

It is obvious from their construction that g_1 and g_2 are homomorphisms. Then, $g_1 \neq g_2$, since $X' \neq X$ and $i(A) \neq X$. We are now going to prove that g_1 and g_2 are homomorphisms of models (i. e. the triangles are commutative) which contradicts the hypothesis that X is an initial model (X is initial $\Longrightarrow$ $g_1 = g_2$). Obviously, $g_1 \circ i = g_2 \circ i = j$. Now, since

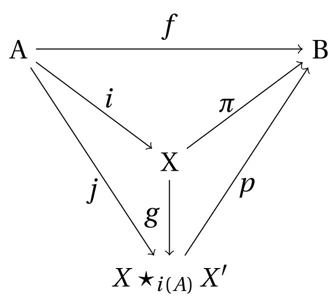
$$p\big(g_1(x)\big) = p(x) = \pi(x)$$

and

$$p\big(g_2(x)\big) = p\big(\underbrace{\phi(x)}_{\in X'}\big) = \pi\big(\phi^{-1}(\phi(x))\big) = \pi(x),$$

we have proved that $g_1 \circ i = g_2 \circ i = j$ and $p \circ g_1 = p \circ g_2 = \pi$.

In conclusion, $g_1 \neq g_2$ and both $g = g_1$ and $g = g_2$ commutatively close the diagram



which contradicts the fact that (X, i, π) is an initial model. The proof is now complete.

□

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4 References

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